King Fahd University of Petroleum & Minerals
College of Computer Sciences & Engineering
Department of Information and Computer Science

ICS 253: Discrete Structures I
Second Exam – 131
90 Minutes

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Question	**Max**	**Earned**
1	22	
2	8	
3	8	
4	7	
5	5	
6	12	
7	15	
8	5	
9	6	
10	6	
11	6	
Total	**100**	

Thursday, November 28, 2013

Key

Question 1: [22 Points] Sets, functions, sequences, and summations
Indicate whether the given sentence is true or false. In the answer column write either ✓ for "true" or × for "false".

Statement	Answer
1. The *universal set* U is the set containing everything currently under consideration.	✓
2. $\emptyset \neq \{\emptyset\}$.	✓
3. A function f from a set A to a set B ($f: A \to B$) is an assignment of one or more element(s) of B to each element of A.	×
4. $A \nsubseteq B$, if $\exists x, x \in A$ with $x \notin B$.	✓
5. If $A \subseteq B$, but $A \neq B$, then A is a *proper subset* of B.	✓
6. A function is *one-to-one* if and only if every element of its range has at least one pre-image.	×
7. The *power set* of A is the set of all subsets of a set A.	✓
8. The union of the sets A and B, $A \cup B$, is the set: $\{x \| x \in A \vee x \in B\}$	✓
9. The complement of $\{x \mid x > 131\}$ is $\{x \mid x \leq 131\}$	✓
10. If $U = \{a,b,c,d,e,f\}$, $A = \{a,b,c,d\}$, $B = \{c,e,f\}$ then $A - B = \{c,e,f\}$.	×
11. In sets $A \cap \emptyset = \emptyset$ and $A \cap A = A$	✓
12. $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$	✓
13. The value of $\lceil \sqrt{5} \rceil$ is 2. (Where $\lceil \quad \rceil$ is the ceiling function).	×
14. If A and B are sets and U is the universal set, then, examples of identity functions are: $A \cup \emptyset$ and $B \cap U$.	✓
15. A string is a finite sequence of characters from a finite set (an alphabet).	✓
16. A geometric progression is a sequence of the form: $a, ar, ar^2, \ldots, ar^n, \ldots$ where the initial term a and the common ratio r are real numbers.	✓
17. An *arithmetic progression* is a sequence of the form: $a, a+d, a+2d, \ldots, a+nd, \ldots$ where the initial term a and the common difference d are real numbers.	✓
18. If x is real and n is integer then $\lfloor x+n \rfloor \neq \lfloor x \rfloor + n$. (Where $\lfloor \quad \rfloor$ is the floor function).	×
19. The nth term of the sequence 5, 6, 10, 28, 124, 724, 5044, … is n! + 4.	✓
20. $\sum_{k=1}^{n}(ak+d) = (a\sum_{k=1}^{n} k) + (nd)$.	✓
21. The set of positive rational numbers is countable.	✓
22. The subset of all real numbers that fall between 0 and 1 is countable.	×

Question 2: [8 Points] Methods of Proof

Prove that if n is an integer, then n^2 is odd if and only if n is odd.

First we show that that n^2 being odd implies that n is odd. We use contrapositive proof.
Suppose n is not odd. Then n is even, so $n = 2a$ for some integer a (by definition of an even number).
Thus $n^2 = (2a)^2 = 2(2a^2)$, so n^2 is even because it's twice an integer. Thus n^2 is not odd.
We have now proved that if n is not odd, then n^2 is not odd, and this is a contrapositive proof that if n^2 is odd then n is odd.

Conversely, we need to prove n being odd implies that n^2 is odd.
Suppose n is odd. Then, by definition of an odd number, $n = 2a + 1$ for some integer a.
Thus $n^2 = (2a+1)^2 = 4a^2+4a+1 = 2(2a^2+2a)+1 = 2k + 1$. (where $k=2a^2+2a$)
This expresses n^2 as twice an integer, plus 1, so n^2 is odd.

Question 3: [8 Points] Methods of Proof

Use proof by contradiction to prove that $1 + \sqrt{2}$ is not rational.
First we prove that the sum of rational number and irrational number is irrational.
Suppose t is irrational and $\frac{a}{b}$ is rational with a, b integers. We want to prove that $t + \frac{a}{b}$ is irrational. Suppose not, then it must be rational, say equal to $\frac{c}{d}$. So we would have $t + \frac{a}{b} = \frac{c}{d}$, so $t = \frac{c}{d} - \frac{a}{b} = \frac{(bc-ad)}{bd}$. This last number has integers for top and bottom so it is rational. This says t is rational. Contradiction. That proves it.

Now we need to prove that $\sqrt{2}$ is irrational.

- Assume that $\sqrt{2}$ is rational
- Then there exists integers a and b with $\sqrt{2} = \frac{a}{b}$ with $b \neq 0$ and a and b having no common factors(i.e. already in lowest terms / reduced form
- square both sides: $2 = \frac{a^2}{b^2}$ which implies $2b^2 = a^2$
- therefore a^2 must be even, if so a must be even
- since a is even, $a = 2c$ for some integer c
- therefore, $2b^2 = (2c)^2 = 4c^2$ which implies $b^2 = 2c^2$
- therefore, b^2 must be even, if so, b must be even
- but then 2 must evenly divide both a and b, which contradicts our assumption that a and b have no common factors.
- We have proven by contradiction that our assumption must be false, and therefore is $\sqrt{2}$ irrational.

Question 4: [7 Points] Methods of Proof

Prove or disprove that if $(a + b)^2$ is even, then both a and b are even.
a = 1, b = 3 then $(a + b)^2 = 4^2 = 16$ which is even, none of a or b is even.
So this counterexample shows that the statement "if $(a + b)^2$ is even, then both a and b are even" is not true.

Question 5: [5 Points] Functions

Determine, with justification, whether *f* is a function from *Z* to *R* if $f(n) = \sqrt{1 - \frac{1}{n^2+1}}$

A Function since the value under the root $0 \leq 1 - \frac{1}{n^2+1} < 1 \; for \; \forall n \; \in \mathbb{Z}$

So $\sqrt{1 - \frac{1}{n^2+1}}$ is a well-defined real number.

Question 6: [12 Points] Cardinality of Sets

(i) Give an example of two uncountable sets A and B such that $A \oplus B$ is empty.
A = [0, 1] B = [0 ,1]

(ii) Give an example of two uncountable sets A and B such that $A \oplus B$ is finite but not empty.
A = [0, 1], B=(0 ,1), A ⊕ B={0, 1}

(iii) Give an example of two uncountable sets A and B such that $A \oplus B$ is countably infinite.

$$A = \bigcup_{i=1}^{\infty} (i, i+1) \qquad B = \bigcup_{i=1}^{\infty} [i, i+1] \qquad A \oplus B = \mathbb{Z}^{+}$$

(iv) Give an example of two uncountable sets A and B such that $A \oplus B$ is uncountable.
A = [0,1), B = [1,2] A ⊕ B = [0,2]

Question 7: [15 Points] Mathematical Induction

Prove that $1.2^0 + 2.2^1 + 3.2^2 + \cdots + n.2^{n-1} = (n-1).2^n + 1$ whenever *n* is a positive integer.

Basis step: We prove for P(1)

Left side: P(1) = 1.2^0 = 1 Right side: P(1) = (1-1).2^1+1 = 1 Therefore 1=1

Hence P(1) proved.

Induction Step: We assume that it is true for P(k)

Summation $1.2^0 + 2.2^1 + 3.2^2 + \cdots + k.2^{k-1} = (k-1).2^k + 1$ is true

We need to prove true for P(k+1)=

$P(k+1) = 1.2^0 + 2.2^1 + 3.2^2 + \cdots + k.2^{k-1} + (k+1).2^k = k.2^{k+1} + 1$.

Add $(k+1).2^k$to both side of the assumption of P(k)

$$P(k+1) = 1.2^0 + 2.2^1 + 3.2^2 + \cdots + k.2^{k-1} + (k+1).2^k$$
$$= (k-1).2^k + 1 + (k+1).2^k = 2^k.\big((k-1) + (k+1)\big) + 1$$
$$= 2^k + 2.k + 1 = k.2^{k+1} + 1$$

Hence proved P(k+1) true for all k>=1, hence

$$1.2^0 + 2.2^1 + 3.2^2 + \cdots + k.2^{k-1} = (k-1).2^k + 1$$

Question 8: [5 Points] Functions

Determine whether $f: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ is onto if $f(m, n) = |m| - |n|$.

If we assume *m* is 0 and run *n* on all values of **Z** we cover all negative integers. If we assume *n* is 0 and run the *m* on all values of **Z** we cover all positive integers. When both *m* and *n* have zero values, the value of the function is 0. So this covers all the values of **Z**. So it is **onto**.

Question 9: [6 Points] Sequences

For the following sequence, identify a rule and the next four terms in the sequence.

2, 2, 2, 3, 5, 5, 7, 7, 11, 11, 11, 11, 13, 13, 17, 17, 17, 17, …

The rule is "the smallest prime number greater than or equal to n."

2, 2, 2, 3, 5, 5, 7, 7, 11, 11, 11, 11, 13, 13, 17, 17, 17, 17, **19, 19, 23, 23,** 23, ..

Question 10: [6 Points] Summations

Use summation notation to write the series 58 + 64 + 70 + ... for 40 terms, then evaluate the sum. Show your steps.

$$\sum_{n=1}^{40}(52+6n)$$

$$\sum_{n=1}^{40}(52+6n) = (40)(52) + 6\sum_{n=1}^{40} n = 2080 + 6\frac{(40)(41)}{2} = 2080 + 4920 = 7000$$

Question 11: [6 Points] Recurrence Relations

Sami earned SR 100,000 during the first year of his job. After each year, he received a 10% raise. Find his total earnings during the first five years on the job. Show your steps.

Let P_n denote the earned amount at year *n*. Because the earned amount after *n* years equals the earned amount after *n-1* years plus the raise for the nth year we see the sequence $\{P_n\}$ satisfies the recurrence relation

$P_n = P_{n-1} + (0.10)\, P_{n-1} = (1.10)\, P_{n-1}$

The initial condition is P_1 = SR 100,000

$P_2 = (1.10)\, P_1$, $P_3 = (1.10)\, P_2 = (1.10)^2 P_1$, $P_4 = (1.10)^3 P_1$, $P_5 = (1.10)^4 P_1$

$P_n = (1.10)\, P_{n-1} = (1.10)^{n-1} P_1$ (A geometric progression where a = 100000 and r = 1.10)

$$\sum_{n=1}^{5}(1.10)^{n-1} P_1 = \frac{((1.10)^5 - 1)\,(100000)}{1.10 - 1.00} = 610510$$

Total earned in the 5 years = SR610510
